

BMO estimates for stochastic singular integral operators and its application to PDEs with Lévy noise

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Abstract

In this paper, we consider the stochastic singular integral operators and obtain the BMO estimates. As an application, we consider the fractional Laplacian equation with additive noises

$$du_t(x) = \Delta^{\frac{\alpha}{2}} u_t(x) dt + \sum_{k=1}^{\infty} \int_{\mathbb{R}^m} g^k(t, x) z \tilde{N}_k(dz, dt), \quad u_0 = 0, \quad 0 \leq t \leq T,$$

where $\Delta^{\frac{\alpha}{2}} = -(-\Delta)^{\frac{\alpha}{2}}$, and $\int_{\mathbb{R}^m} z \tilde{N}_k(t, dz) =: Y_t^k$ are independent m -dimensional pure jump Lévy processes with Lévy measure of ν^k . Following the idea of [9], we obtain the q -th order BMO quasi-norm of the $\frac{\alpha}{q_0}$ -order derivative of u is controlled by the norm of g .

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1 Introduction

For a stochastic process $\{X_t, t \in T\}$, there are two important facts worth studying. One is its probability density function (PDF) or its probability law, the other is the estimates of moment. But for a stochastic process depending on spatial variable, that is, $X_t = X(t, \omega, x)$ (x is the spatial variable), it is hard to consider its PDF or probability law. Fortunately, we can get some estimates of moment. In this paper, we focus on the estimates of solutions of stochastic partial differential equations (SPDEs).

For SPDEs, many kinds of estimates of the solutions have been well studied. By using parabolic Littlewood-Paley inequality, Krylov [13] proved that for SPDEs of the type

$$du = \Delta u dt + g dw_t, \tag{1.1}$$

it holds that

$$\mathbb{E}\|\nabla u\|_{L^p((0,T)\times\mathbb{R}^d)}^p \leq C(d,p)\mathbb{E}\|g\|_{L^p((0,T)\times\mathbb{R}^d)}^p, \quad (1.2)$$

where w_t is a Wiener process and $p \in [2, \infty)$. van Neerven et al. [16] introduce a significant extension of (1.2) to a class of operators A which admit a bounded H^∞ -calculus of angle less than $\pi/2$. Kim [9] established a BMO estimate for stochastic singular integral operators. And as an application, they considered (1.1) and obtained the q -th order BMO quasi-norm of the derivative of u is controlled by $\|g\|_{L^\infty}$. Just recently, Kim et al. [11] studied the parabolic Littlewood-Paley inequality for a class of time-dependent pseudo-differential operators of arbitrary order, and applied this result to the high-order stochastic PDE.

Recently, Yang [18] considered the following SPDEs

$$du = \Delta^{\frac{\alpha}{2}} u dt + f dX_t, \quad u_0 = 0, \quad 0 < t < T,$$

where $\Delta^{\frac{\alpha}{2}} = -(-\Delta)^{\frac{\alpha}{2}}$, $0 < \alpha < 2$, and X_t is a Lévy process. They obtained a parabolic Triebel-Lizorkin space estimate for the convolution operator.

Regarding elliptic and parabolic singular integral operators, the BMO estimates was already established in [4, 6]. In this paper, we consider the stochastic singular integral operator

$$\begin{aligned} \mathcal{G}g(t, x) &= \int_0^t \int_Z K(t, s, \cdot) * g(s, \cdot, z)(x) \tilde{N}(dz, ds) \\ &= \int_0^t \int_Z \int_{\mathbb{R}^d} K(t, s, x - y) g(s, y, z) dy \tilde{N}(dz, ds). \end{aligned} \quad (1.3)$$

Our main purpose is to present appropriate conditions on the kernel K for the following estimate:

$$\begin{aligned} [\mathcal{G}g]_{\mathbb{BMO}(T,q)} &\leq N \left(\left\| \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{q/2} \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right. \\ &\quad + \left\| \int_Z |g(\cdot, \cdot, z)|_{L^\infty(\mathcal{O}_T)}^{q_0} \nu(dz) \right\|_{L^{\tilde{\kappa}}(\Omega)}^{q/q_0} \\ &\quad \left. + \left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right), \end{aligned} \quad (1.4)$$

where $q \in [2, p_0 \wedge \kappa]$, $\tilde{\kappa}$ is the conjugate of a positive constant κ , the constant N depends on q and d , and ν is a measure, see Section 2. As an application of (1.4), we prove that the solution of the following equation

$$du_t(x) = \Delta^{\frac{\alpha}{2}} u_t(x) dt + \sum_{k=1}^{\infty} \int_{\mathbb{R}^m} g^k(t, x) z \tilde{N}_k(dz, dt), \quad u_0 = 0, \quad 0 \leq t \leq T,$$

satisfies that for $q \in [2, q_0]$

$$[\nabla^\beta u]_{\mathbb{BMO}(T,q)} \leq N \hat{c} \left(\mathbb{E}[\|g\|_{\ell_2}^{q_0}]_{L^\infty(\mathcal{O}_T)} \right)^{q/q_0},$$

where $\int_{\mathbb{R}^m} z \tilde{N}_k(t, dz) =: Y_t^k$ are independent m -dimensional pure jump Lévy processes with Lévy measure of ν^k , $\beta = \alpha/q_0$ and \hat{c} is defined as in (4.4), see Section 4 for details. Moreover, we find if we consider the following stochastic parabolic equation

$$du_t(x) = \Delta^{\frac{\alpha}{2}} u_t(x) dt + \sum_{k=1}^{\infty} h^k(t, x) dW_t^k, \quad u_0 = 0, \quad 0 \leq t \leq T,$$

where W_t^k are independent one-dimensional Wiener processes. We have the following estimate, for any $q \in (0, p]$,

$$[\nabla^{\frac{\alpha}{2}} u]_{\mathbb{BMO}(T, q)} \leq N \left(\mathbb{E}[\|h\|_{\ell_2}^p] \right)^{1/p}.$$

under the condition that $h \in L^p(T, \ell_2)$, see Theorem 4.2. Specially, taking $\alpha = 2$, we obtain the result of [9, Theorem 3.4].

Due to the difference between the Brownian motion and Lévy process, it is more difficult to get the BMO estimate for Lévy process. Following the idea of [9], we obtain the BMO estimate of stochastic singular integral operators. We remark that there are many places different from those in [9]. First, the assumptions on the kernel are different from those in [9], see Section 2; Second, the exponent q in [9] do not depend on the properties of kernel but we do. For simplicity, we only consider a simple case, see the discussion in Section 4.

This paper is organized as follows. In Section 2, we introduce the main results. The proof of the main results is complete in section 3. Section 4 is concerned with an application of our result. This paper ends with a short discussion, which shows that we can give a simple proof of the result in Section 2 if the function g has high regularity.

Before we end this section, we introduce some notations used in this paper. As usual \mathbb{R}^d stands for the Euclidean space of points $x = (x_1, \dots, x_d)$, $B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}$ and $B_r := B_r(0)$. \mathbb{R}_+ denotes the set $\{x \in \mathbb{R}, x > 0\}$. $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$ and $L^p := L^p(\mathbb{R}^d)$. $N = N(a, b, \dots)$ means that the constant N depends only on a, b, \dots .

2 Known results and Main result

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space such that \mathcal{F}_t is a filtration on Ω containing all P -null subsets of Ω and \mathbb{F} be the predictable σ -field by $(\mathcal{F}_t, t \geq 0)$. We are given a measure space (Z, \mathcal{Z}, ν) and a Poisson measure μ on $[0, T] \times Z$, defined on the stochastic basis. The compensator of μ is $\text{Leb} \otimes \nu$, and the compensated measure $\tilde{N} := \mu - \text{Leb} \otimes \nu$

Fix $\gamma > 0$ and $T \in (0, \infty]$. Denote

$$\mathcal{O}_T = (0, T) \times \mathbb{R}^d.$$

For a measurable function h on $\Omega \times \mathcal{O}_T$, we define the q -th order stochastic BMO (Bounded mean oscillation) quasi-norm of h on $\Omega \times \mathcal{O}_T$ as follows:

$$[h]_{\mathbb{BMO}(T, q)}^q = \sup_Q \frac{1}{Q^2} \mathbb{E} \int_Q \int_Q |h(t, x) - h(s, y)|^q dt dx ds dy,$$

where the sup is taken over all Q of the type

$$Q = Q_c(t_0, x_0) := (t_0 - c^\gamma, t_0 + c^\gamma) \times B_c(x_0) \subset \mathcal{O}_T, \quad c > 0, t_0 > 0.$$

It is remarked that when $q = 1$, this is equivalent to the classical BMO semi-norm which is introduced by John-Nirenberg [8].

Let $K(\omega, t, s, x)$ be a measurable function on $\Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d$ such that for each $t \in \mathbb{R}_+$, $(\omega, s) \mapsto K(\omega, t, s, \cdot)$ is a predictable L_{loc}^1 -valued process.

Firstly, we recall the results of [9]. In [9], the following assumptions are needed.

Assumption 2.1 There exist a $\kappa \in [1, \infty]$ and a nondecreasing function $\varphi(t) : (0, \infty) \mapsto [0, \infty)$ such that

(i) for any $t > \lambda > 0$ and $c > 0$,

$$\left\| \int_{\lambda}^t \left| \int_{|x| \geq c} |K(t, r, x)| dx \right|^2 dr \right\|_{L^{\kappa/2}(\Omega)} \leq \varphi((t - \lambda)c^{-\gamma});$$

(ii) for any $t > s > \lambda > 0$,

$$\left\| \int_0^{\lambda} \left(\int_{\mathbb{R}^d} |K(t, r, x) - K(s, r, x)| dx \right)^2 dr \right\|_{L^{\kappa/2}(\Omega)} \leq \varphi((t - s)(t \wedge s - \lambda)^{-1});$$

(iii) for any $s > \lambda \geq 0$ and $h \in \mathbb{R}^d$,

$$\left\| \int_0^{\lambda} \left(\int_{\mathbb{R}^d} |K(s, r, x + h) - K(s, r, x)| dx \right)^2 dr \right\|_{L^{\kappa/2}(\Omega)} \leq N\varphi(|h|(s - \lambda)^{-1/\gamma}).$$

Assumption 2.2 Suppose that $\mathcal{G}g$ is well-defined (a.e.) and the following holds:

$$\mathbb{E} \int_0^T \|\mathcal{G}g(t, \cdot)\|_{L^{p_0}}^{p_0} dt \leq N_0 \left\| \int_0^T \|g(t, \cdot)\|_{l_2}^{p_0} dt \right\|_{L^{\tilde{\kappa}}(\Omega)},$$

where $\tilde{\kappa}$ is the conjugate of κ , and

$$\mathcal{G}g(t, x) = \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}^d} K(t, s, x - y) g^k(s, y) dy dw_s^k,$$

with w_t is a Wiener process. Under the Assumptions 2.1 and 2.2, Kim obtained the BMO estimate of $\mathcal{G}g$.

Comparing with the assumption 2.1, due to the Kunita's first inequality (see Page 265 of [1]), we need the following assumptions. For the Kunita's inequality and BDG inequality of Lévy noise, see Lemma 3.1 of [14] and [15] respectively.

Assumption 2.3 There exist constants $q_0 \geq 2$, $\kappa \in [1, \infty]$ and a nondecreasing function $\varphi(t) : (0, \infty) \mapsto [0, \infty)$ such that

(i) for any $t > \lambda > 0$ and $c > 0$,

$$\left\| \int_{\lambda}^t \left| \int_{|x| \geq c} |K(t, r, x)| dx \right|^{q_0} dr \right\|_{L^{\kappa/q_0}(\Omega)} \leq \varphi((t - \lambda)c^{-\gamma});$$

(ii) for any $t > s > \lambda > 0$,

$$\left\| \int_0^{\lambda} \left(\int_{\mathbb{R}^d} |K(t, r, x) - K(s, r, x)| dx \right)^{q_0} dr \right\|_{L^{\kappa/q_0}(\Omega)} \leq \varphi((t - s)(t \wedge s - \lambda)^{-1});$$

(iii) for any $s > \lambda \geq 0$ and $h \in \mathbb{R}^d$,

$$\left\| \int_0^{\lambda} \left(\int_{\mathbb{R}^d} |K(s, r, x + h) - K(s, r, x)| dx \right)^{q_0} dr \right\|_{L^{\kappa/q_0}(\Omega)} \leq N\varphi(|h|(s - \lambda)^{-1/\gamma}).$$

Remark 2.1 The difference between assumptions 2.1 and 2.3 is because the following Kunita's first inequality.

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq t \leq T} |I(t)|^p \right) &\leq N(p) \left\{ \mathbb{E} \left[\left(\int_0^T \int_Z |H(t, z)|^2 \nu(dz) dt \right)^{p/2} \right] \right. \\ &\quad \left. + \mathbb{E} \left[\int_0^T \int_Z |H(t, z)|^p \nu(dz) dt \right] \right\}, \end{aligned} \quad (2.1)$$

where $p \geq 2$ and

$$I(t) = \int_0^t \int_Z H(s, z) \tilde{N}(dz, ds).$$

When $\tilde{N}(ds, dz)$ is replaced by $dw_s dz$, the second term of right side hand of (2.1) will disappear. Hence, in order to deal with the difficult from the Lévy process, we give the assumption 2.3.

Assumption 2.4 Similar Assumption 2.2, suppose that $\mathcal{G}g$ is well-defined (a.e.) and the following holds:

$$\mathbb{E} \int_0^T \|\mathcal{G}g(t, \cdot)\|_{L^{q_0}}^{q_0} dt \leq N_0 \left\| \int_0^T \int_Z \|g(t, \cdot, z)\|_{L^{q_0}}^{q_0} \nu(dz) dt \right\|_{L^{\tilde{\kappa}}(\Omega)}. \quad (2.2)$$

Our main result is the following.

Theorem 2.1 Let Assumptions 2.3 and 2.4 hold. Assume that the function g satisfies

$$\left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^{\varpi} \nu(dz) \right\|_{L^\varsigma(\Omega)} < \infty, \quad \varpi = 2 \text{ or } q_0, \quad (2.3)$$

where $\varsigma = q_0 \tilde{\kappa} \vee \frac{q_0 \kappa}{2(\kappa - q_0)^+}$ ($\varsigma = \infty$ if $\kappa \leq q_0$). Then for any $q \in [2, q_0 \wedge \kappa]$, one has

$$\begin{aligned} [\mathcal{G}g]_{\mathbb{BMO}(T, q)} &\leq N \left(\left\| \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{q/2} \right\|_{L^{\frac{\kappa}{\kappa - q}}} \right. \\ &\quad + \left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^{q_0} \nu(dz) \right\|_{L^{\tilde{\kappa}}(\Omega)}^{q/q_0} \\ &\quad \left. + \left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right\|_{L^{\frac{\kappa}{\kappa - q}}} \right), \end{aligned} \quad (2.4)$$

where $N = N(N_0, d, q, q_0, \gamma, \kappa, \varphi)$.

Remark 2.2 1. Comparing Theorem 2.1 with Theorem 2.4 in [9], it is not hard to find in Theorem 2.4 of [9] the exponent q does not depend on q_0 . Actually, the range of exponent q is $(0, p_0 \wedge \kappa]$ and in this paper is $[2, q_0 \wedge \kappa]$. In other words, the range of exponent q depends on the properties of kernel K . The lower bound of q is because the Kunita's first inequality holds for $q \geq 2$.

2. In Theorem 2.1, we did not write the right hand of (2.4) as a uniform format. The reason is that $\int_Z \nu(dz)$ maybe not exist. If we assume that

$$\int_Z (z^2 \wedge 1) \nu(dz) \leq N_1 \text{ and } \left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^{q_0} (1 + f(z)^{-\frac{q_0}{2}} + f(z)^{-\frac{q_0}{q}}) \nu(dz) \right\|_{L^{\kappa^*}(\Omega)} < \infty,$$

where N_1 is a positive constant, then (2.4) can be replaced by

$$[\mathcal{G}g]_{\mathbb{BMO}(T, q)} \leq \left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^{q_0} (1 + f(z)^{-\frac{q_0}{2}} + f(z)^{-\frac{q_0}{q}}) \nu(dz) \right\|_{L^{\kappa^*}(\Omega)}^{q/q_0},$$

where

$$\kappa^* = \tilde{\kappa} \vee \frac{\kappa}{\kappa - q}, \quad f(z) = \frac{z^2 + 1 - |z^2 - 1|}{2} = z^2 \wedge 1.$$

3. The condition (2.3) coincides with (4.4) in Section 4. Under the condition (2.3), it is easy to check that

$$\left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right\|_{L^{\frac{\kappa}{\kappa - q}}(\Omega)} < \infty.$$

3 Proof of the main result

In this section, we first estimate the expectation of local mean average of $\mathcal{G}g$ and its difference in terms of the supremum of $|g|$ given a vanishing condition on g . Then we complete the proof of main result.

Lemma 3.1 *Let $q \in [2, q_0]$, $0 \leq a \leq b \leq T$, and Assumption 2.4 hold. Suppose that g vanishes on $(a, b) \times (B_{3c})^c \times Z$ and $(0, a) \times \mathbb{R}^d \times Z$. Then*

$$\mathbb{E} \int_a^b \int_{B_c} |\mathcal{G}g(t, x)|^q dx dt \leq N(b-a)|B_{3c}| \left\| \sup_{(a,b) \times B_{3c}} \int_Z |g(\cdot, \cdot, z)|^{q_0} \nu(dz) \right\|_{L^{\tilde{\kappa}}(\Omega)}^{q/p_0},$$

where $N = N(N_0)$.

Proof. The proof of this lemma is similar to that of Lemma 4.1 in [9]. In order to read easily, we give the outline of the proof. By Hölder's inequality and Assumption 2.4,

$$\begin{aligned} & \mathbb{E} \int_a^b \int_{B_c} |\mathcal{G}g(t, x)|^q dx dt \\ & \leq (b-a)^{(q_0-q)/q_0} |B_c|^{(q_0-q)/q_0} \left(\mathbb{E} \int_a^b \int_{B_c} |\mathcal{G}g(t, x)|^{q_0} dx dt \right)^{q/q_0} \\ & \leq N(b-a)^{(q_0-q)/q_0} |B_c|^{(q_0-q)/q_0} \left\| \int_0^T \int_Z \|g(t, \cdot, z)\|_{L^{q_0}}^{q_0} \nu(dz) dt \right\|_{L^{\tilde{\kappa}}(\Omega)}^{q/q_0}. \end{aligned}$$

Since g vanishes on $(a, b) \times (B_{3c})^c$ and $(0, a) \times \mathbb{R}^d$, the above term is equal to or less than

$$\begin{aligned} & N(b-a)^{(q_0-q)/q_0} |B_c|^{(q_0-q)/q_0} \left\| \int_a^b \int_{B_{3c}} \int_Z |g(t, x, z)|^{q_0} \nu(dz) dx dt \right\|_{L^{\tilde{\kappa}}(\Omega)}^{q/q_0} \\ & \leq N(b-a)|B_{3c}| \left\| \sup_{(a,b) \times B_{3c}} \int_Z |g(\cdot, \cdot, z)|^{q_0} \nu(dz) \right\|_{L^{\tilde{\kappa}}(\Omega)}^{q/q_0}. \end{aligned}$$

The proof of lemma is complete. \square

Lemma 3.2 *Let $q \in [2, q_0 \wedge \kappa]$, $0 \leq a \leq b \leq T$ and Assumption 2.3 (i) hold. Suppose that g vanishes on $(0, \frac{3b-a}{2}) \times B_{2c} \times Z$. Then*

$$\begin{aligned} & \mathbb{E} \int_a^b \int_{B_c} \int_a^b \int_{B_c} |\mathcal{G}g(t, x) - \mathcal{G}g(s, y)|^q dx dt ds dy \\ & \leq N(b-a)^2 |B_c|^2 [\varphi(bc^{-\gamma})]^{q/q_0} \left(\left\| \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{q/2} \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right. \\ & \quad \left. + \left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right), \end{aligned} \tag{3.1}$$

where $\frac{\infty}{\infty} := 1$ and $N = N(T, q)$.

Proof. Let $(t, x) \in (a, b) \times B_c$ and $0 \leq r \leq t$. If $|y| \leq c$, then $(r, x-y) \in (0, \frac{3b-a}{2}) \times B_{2c}$ and $g(r, x-y, z) = 0$ for all $z \in Z$. Hence, Assumption 2.3 (i), Hölder inequality and Kunita's first

inequality (2.1) implies

$$\begin{aligned}
\mathbb{E}|\mathcal{G}g(t, x)|^q &\leq \mathbb{E} \left(\int_0^t \int_Z \left| \int_{\mathbb{R}^d} K(t, r, y) g(r, x - y, z) dy \right|^2 \nu(dz) dr \right)^{q/2} \\
&\quad + \mathbb{E} \left(\int_0^t \int_Z \left| \int_{\mathbb{R}^d} K(t, r, y) g(r, x - y, z) dy \right|^q \nu(dz) dr \right) \\
&\leq \mathbb{E} \left(\int_0^t \int_Z \left| \int_{|y| \geq c} K(t, r, y) g(r, x - y, z) dy \right|^2 \nu(dz) dr \right)^{q/2} \\
&\quad + \mathbb{E} \left(\int_0^t \int_Z \left| \int_{|y| \geq c} K(t, r, y) g(r, x - y, z) dy \right|^q \nu(dz) dr \right) \\
&\leq T^{(q_0-2)q/(2q_0)} \mathbb{E} \left[\left(\int_0^t \left| \int_{|y| \geq c} K(t, r, y) dy \right|^{q_0} dr \right)^{q/q_0} \right. \\
&\quad \left. \times \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{q/2} \right] \\
&\quad + \mathbb{E} \left[\left(\int_0^t \left| \int_{|y| \geq c} K(t, r, y) dy \right|^q dr \right) \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right] \\
&\leq N(T) \left\| \int_0^t \left| \int_{|y| \geq c} K(t, r, y) dy \right|^{q_0} dr \right\|_{L^{\kappa/q_0}}^{q/q_0} \\
&\quad \times \left\| \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{q/2} \right\|_{L^{\frac{\kappa}{\kappa-q}}} \\
&\quad + N(T) \left\| \int_0^t \left| \int_{|y| \geq c} K(t, r, y) dy \right|^{q_0} dr \right\|_{L^{\kappa/q_0}}^{q/q_0} \\
&\quad \times \left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right\|_{L^{\frac{\kappa}{\kappa-q}}} \\
&\leq N(T) [\varphi(bc^{-\gamma})]^{q/q_0} \left(\left\| \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{q/2} \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right. \\
&\quad \left. + \left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\mathbb{E} \int_a^b \int_{B_c} \int_a^b \int_{B_c} |\mathcal{G}g(t, x) - \mathcal{G}g(s, y)|^q dx dt ds dy \\
&\leq N(q)(b-a)|B_c| \mathbb{E} \int_a^b \int_{B_c} |\mathcal{G}g(t, x)|^q dx dt \\
&\leq N(T, q)(b-a)^2 |B_c|^2 [\varphi(bc^{-\gamma})]^{q/q_0} \left(\left\| \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{q/2} \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right. \\
&\quad \left. + \left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right).
\end{aligned}$$

The inequality (3.1) is obtained. The proof of lemma is complete. \square

Lemma 3.3 *Let $q \in [2, q_0 \wedge \kappa]$, $0 \leq a < b \leq T$ such that $3a > b$. Suppose that Assumption 2.3 holds and $\mathcal{G}g$ is well-defined almost everywhere. Assume further that g vanishes on $(\frac{3a-b}{2}, \frac{3b-a}{2}) \times B_{2c} \times Z$. Then*

$$\begin{aligned} & \mathbb{E} \int_a^b \int_{B_c} \int_a^b \int_{B_c} |\mathcal{G}g(t, x) - \mathcal{G}g(s, y)|^q dx dt ds dy \\ & \leq N(b-a)^2 |B_c|^2 \Phi(a, b, c) \left(\left\| \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{q/2} \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right. \\ & \quad \left. + \left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right), \end{aligned} \quad (3.2)$$

where $N = N(T, q, a, b, c)$ and

$$\Phi(a, b, c) = [\varphi(2)]^{q/q_0} + [\varphi((b-a)c^{-\gamma})]^{q/q_0} + [\varphi(2^{1+1/\gamma}c(b-a)^{-1/\gamma})]^{q/q_0}.$$

Proof. Due to the Fubini's Theorem, it suffices to prove that for all $(t, x) \in (a, b) \times B_c$ and $(s, y) \in (a, b) \times B_c$, the following inequality holds:

$$\begin{aligned} \mathbb{E} |\mathcal{G}g(t, x) - \mathcal{G}g(s, y)|^q & \leq N \Phi(a, b, c) \left(\left\| \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{q/2} \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right. \\ & \quad \left. + \left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right), \end{aligned}$$

Obviously,

$$\begin{aligned} & \mathbb{E} |\mathcal{G}g(t, x) - \mathcal{G}g(s, y)|^q \\ & \leq N(\mathbb{E} |\mathcal{G}g(t, x) - \mathcal{G}g(s, x)|^q + \mathbb{E} |\mathcal{G}g(s, x) - \mathcal{G}g(s, y)|^q) \\ & =: N(I_1 + I_2). \end{aligned}$$

Estimate of I_1 . Without loss of generality we assume $t \geq s$. Hence by Lemma 3.1 of [14] and

(2.1), we get

$$\begin{aligned}
I_1 &= \mathbb{E} |\mathcal{G}g(t, x) - \mathcal{G}g(s, x)|^q \\
&= \mathbb{E} \left[\left| \int_0^t \int_Z \int_{\mathbb{R}^d} K(t, r, x - y) g(r, y, z) dy \tilde{N}(dz, dr) \right. \right. \\
&\quad \left. \left. - \int_0^s \int_Z \int_{\mathbb{R}^d} K(s, r, x - y) g(r, y, z) dy \tilde{N}(dz, dr) \right|^q \right] \\
&\leq N \mathbb{E} \left[\left| \int_0^t \int_Z \int_{\mathbb{R}^d} K(t, r, x - y) g(r, y, z) dy \tilde{N}(dz, dr) \right. \right. \\
&\quad \left. \left. - \int_0^s \int_Z \int_{\mathbb{R}^d} K(t, r, x - y) g(r, y, z) dy \tilde{N}(dz, dr) \right|^q \right] \\
&\quad + N \mathbb{E} \left[\left| \int_0^s \int_Z \int_{\mathbb{R}^d} (K(t, r, x - y) - K(s, r, x - y)) g(r, y, z) dy \tilde{N}(dz, dr) \right|^q \right] \\
&\leq N \mathbb{E} \left[\left(\int_s^t \int_Z \left| \int_{\mathbb{R}^d} K(t, r, x - y) g(r, y, z) dy \right|^2 \nu(dz) dr \right)^{q/2} \right] \\
&\quad + N \mathbb{E} \left[\int_s^t \int_Z \left| \int_{\mathbb{R}^d} K(t, r, x - y) g(r, y, z) dy \right|^q \nu(dz) dr \right] \\
&\quad + N \mathbb{E} \left[\left(\int_0^s \int_Z \left| \int_{\mathbb{R}^d} (K(t, r, x - y) - K(s, r, x - y)) g(r, y, z) dy \right|^2 \nu(dz) dr \right)^{q/2} \right] \\
&\quad + N \mathbb{E} \left[\int_0^s \int_Z \left| \int_{\mathbb{R}^d} (K(t, r, x - y) - K(s, r, x - y)) g(r, y, z) dy \right|^q \nu(dz) dr \right] \\
&=: N(I_{11} + I_{12} + I_{13} + I_{14}).
\end{aligned}$$

Note that g vanishes on $(\frac{3a-b}{2}, \frac{3b-a}{2}) \times B_{2c} \times Z$ and $a > \frac{3a-b}{2}$. **Assumption 2.3 (i)** with $\lambda = s$ yields that

$$\begin{aligned}
I_{11} + I_{12} &= \mathbb{E} \left[\left(\int_s^t \int_Z \left| \int_{\mathbb{R}^d} K(t, r, y) g(r, x - y, z) dy \right|^2 \nu(dz) dr \right)^{q/2} \right] \\
&\quad + \mathbb{E} \left[\int_s^t \int_Z \left| \int_{\mathbb{R}^d} K(t, r, y) g(r, x - y, z) dy \right|^q \nu(dz) dr \right] \\
&\leq \mathbb{E} \left[\left(\int_s^t \left| \int_{|y| \geq c} |K(t, r, y)| dy \right|^2 dr \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{q/2} \right] \\
&\quad + \mathbb{E} \left[\int_s^t \left| \int_{|y| \geq c} |K(t, r, y)| dy \right|^q \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) dt \right] \\
&\leq N[\varphi((b-a)c^{-\gamma})]^{q/q_0} \left(\left\| \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{q/2} \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right. \\
&\quad \left. + \left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right).
\end{aligned}$$

Similarly, due to g vanishes on $(\frac{3a-b}{2}, \frac{3b-a}{2}) \times B_{2c} \times Z$, we divide $(0, s)$ into two parts $(0, \frac{3a-b}{2})$ and

$(\frac{3a-b}{2}, s)$. And thus we have

$$\begin{aligned}
I_{13} + I_{14} &= \mathbb{E} \left[\left(\int_{\frac{3a-b}{2}}^s \int_Z \left| \int_{\mathbb{R}^d} (K(t, r, x-y) - K(s, r, x-y)) g(r, y, z) dy \right|^2 \nu(dz) dr \right)^{q/2} \right] \\
&+ \mathbb{E} \left[\int_{\frac{3a-b}{2}}^s \int_Z \left| \int_{\mathbb{R}^d} (K(t, r, x-y) - K(s, r, x-y)) g(r, y, z) dy \right|^q \nu(dz) dr \right] \\
&+ \mathbb{E} \left[\left(\int_0^{\frac{3a-b}{2}} \int_Z \left| \int_{\mathbb{R}^d} (K(t, r, x-y) - K(s, r, x-y)) g(r, y, z) dy \right|^2 \nu(dz) dr \right)^{q/2} \right] \\
&+ \mathbb{E} \left[\int_0^{\frac{3a-b}{2}} \int_Z \left| \int_{\mathbb{R}^d} (K(t, r, x-y) - K(s, r, x-y)) g(r, y, z) dy \right|^q \nu(dz) dr \right] \\
&=: I_{131} + I_{141} + I_{132} + I_{142}.
\end{aligned}$$

Using again **Assumption 2.3 (i)** with $\lambda = \frac{3a-b}{2}$, we get

$$\begin{aligned}
I_{131} + I_{141} &\leq \mathbb{E} \left[\left(\int_{\frac{3a-b}{2}}^t \int_Z \left| \int_{\mathbb{R}^d} K(t, r, x-y) g(r, y, z) dy \right|^2 \nu(dz) dr \right)^{q/2} \right] \\
&+ \mathbb{E} \left[\left(\int_{\frac{3a-b}{2}}^s \int_Z \left| \int_{\mathbb{R}^d} K(s, r, x-y) g(r, y, z) dy \right|^2 \nu(dz) dr \right)^{q/2} \right] \\
&+ \mathbb{E} \left[\int_{\frac{3a-b}{2}}^t \int_Z \left| \int_{\mathbb{R}^d} K(t, r, x-y) g(r, y, z) dy \right|^q \nu(dz) dr \right] \\
&+ \mathbb{E} \left[\int_{\frac{3a-b}{2}}^s \int_Z \left| \int_{\mathbb{R}^d} K(s, r, x-y) g(r, y, z) dy \right|^q \nu(dz) dr \right] \\
&\leq N[\varphi(2(b-a)c^{-\gamma})]^{q/q_0} \left(\left\| \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{q/2} \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right. \\
&\quad \left. + \left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right).
\end{aligned}$$

On the other hand, **Assumption 2.3 (ii)** with $\lambda = \frac{3a-b}{2}$ gives

$$\begin{aligned}
I_{132} + I_{142} &\leq N \mathbb{E} \left[\left(\int_0^{\frac{3a-b}{2}} \left| \int_{\mathbb{R}^d} |K(t, r, x-y) - K(s, r, x-y)| dy \right|^2 dr \right. \right. \\
&\quad \left. \left. \times \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{q/2} \right] \\
&+ \mathbb{E} \left[\int_0^{\frac{3a-b}{2}} \left| \int_{\mathbb{R}^d} |K(t, r, x-y) - K(s, r, x-y)| dy \right|^q \right. \\
&\quad \left. \times \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) dr \right] \\
&\leq N[\varphi(2)]^{q/q_0} \left(\left\| \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{q/2} \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right. \\
&\quad \left. + \left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right),
\end{aligned}$$

where we used $s - \frac{3a-b}{2} \geq a - \frac{3a-b}{2} = \frac{b-a}{2}$ and $(t-s)(s - \frac{3a-b}{2})^{-1} \leq 2$.

Estimate of I_2 . By using the fact $g = 0$ on $(\frac{3a-b}{2}, \frac{3b-a}{2}) \times B_{2c} \times Z$ again, we divide $(0, s)$ into two parts $(0, \frac{3a-b}{2})$ and $(\frac{3a-b}{2}, s)$. Direct calculations shows that

$$\begin{aligned}
I_2 &\leq N\mathbb{E} \left(\int_0^s \int_Z \left| \int_{\mathbb{R}^d} K(s, r, w)(g(r, x - w, z) - g(r, y - w, z))dw \right|^2 \nu(dz)dr \right)^{q/2} \\
&\quad + \mathbb{E} \left(\int_0^s \int_Z \left| \int_{\mathbb{R}^d} K(s, r, w)(g(r, x - w, z) - g(r, y - w, z))dw \right|^q \nu(dz)dr \right) \\
&\leq N\mathbb{E} \left(\int_{\frac{3a-b}{2}}^s \int_Z \left| \int_{\mathbb{R}^d} K(s, r, w)(g(r, x - w, z) - g(r, y - w, z))dw \right|^2 \nu(dz)dr \right)^{q/2} \\
&\quad + N\mathbb{E} \left(\int_{\frac{3a-b}{2}}^s \int_Z \left| \int_{\mathbb{R}^d} K(s, r, w)(g(r, x - w, z) - g(r, y - w, z))dw \right|^q \nu(dz)dr \right) \\
&\quad + N\mathbb{E} \left(\int_0^{\frac{3a-b}{2}} \int_Z \left| \int_{\mathbb{R}^d} (K(s, r, x - w) - K(s, r, y - w))g(r, w, z)dw \right|^2 \nu(dz)dr \right)^{q/2} \\
&\quad + N\mathbb{E} \left(\int_0^{\frac{3a-b}{2}} \int_Z \left| \int_{\mathbb{R}^d} (K(s, r, x - w) - K(s, r, y - w))g(r, w, z)dw \right|^q \nu(dz)dr \right) \\
&\leq N\mathbb{E} \left(\int_{\frac{3a-b}{2}}^s \int_Z \left| \int_{\mathbb{R}^d} |K(s, r, w)g(r, x - w, z)|dw \right|^2 \nu(dz)dr \right)^{q/2} \\
&\quad + N\mathbb{E} \left(\int_{\frac{3a-b}{2}}^s \int_Z \left| \int_{\mathbb{R}^d} |K(s, r, w)g(r, y - w, z)|dw \right|^2 \nu(dz)dr \right)^{q/2} \\
&\quad + N\mathbb{E} \left(\int_{\frac{3a-b}{2}}^s \int_Z \left| \int_{\mathbb{R}^d} |K(s, r, w)g(r, x - w, z)|dw \right|^q \nu(dz)dr \right) \\
&\quad + N\mathbb{E} \left(\int_{\frac{3a-b}{2}}^s \int_Z \left| \int_{\mathbb{R}^d} |K(s, r, w)g(r, y - w, z)|dw \right|^q \nu(dz)dr \right) \\
&\quad + N\mathbb{E} \left(\int_0^{\frac{3a-b}{2}} \int_Z \left| \int_{\mathbb{R}^d} (K(s, r, x - w) - K(s, r, y - w))g(r, w, z)dw \right|^2 \nu(dz)dr \right)^{q/2} \\
&\quad + N\mathbb{E} \left(\int_0^{\frac{3a-b}{2}} \int_Z \left| \int_{\mathbb{R}^d} (K(s, r, x - w) - K(s, r, y - w))g(r, w, z)dw \right|^q \nu(dz)dr \right) \\
&=: I_{21} + \cdots + I_{26}.
\end{aligned}$$

Similar to $I_{11} + I_{12}$, the four terms $I_{21} + \cdots + I_{24}$ is less than or equal to

$$\begin{aligned}
&N[\varphi(2(b-a)c^{-\gamma})]^{q/q_0} \left(\left\| \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{q/2} \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right. \\
&\quad \left. + \left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right).
\end{aligned}$$

Using **Assumption 2.3 (iii)** with $\lambda = \frac{3a-b}{2}$, we get

$$\begin{aligned}
I_{25} + I_{26} &\leq N \mathbb{E} \left(\int_0^{\frac{3a-b}{2}} \left| \int_{\mathbb{R}^d} |K(s, r, x-w) - K(s, r, y-w)| dw \right|^2 dr \right. \\
&\quad \times \left. \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{q/2} \\
&\quad + N \mathbb{E} \left(\int_0^{\frac{3a-b}{2}} \left| \int_{\mathbb{R}^d} |K(s, r, x-w) - K(s, r, y-w)| dw \right|^q dr \right. \\
&\quad \times \left. \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right) \\
&\leq N \varphi(2^{1+1/\gamma} c(b-a)^{-1/\gamma})^{q/q_0} \left(\left\| \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{q/2} \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right. \\
&\quad \left. + \left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right).
\end{aligned}$$

Combining the above discussion, (3.2) is obtained. The proof of this lemma is complete. \square

Now, we are ready to prove the main result. The proof is similar to that of Theorem of 2.4 in [9].

Proof of Theorem 2.1. Let $q \in [2, q_0 \wedge \kappa]$. It suffices to prove that for each

$$Q = Q_c(t_0, x_0) := (t_0 - c^\gamma, t_0 + c^\gamma) \times B_c(x_0) \subset \mathcal{O}_T, \quad c > 0, t_0 > 0,$$

we have

$$\begin{aligned}
&\frac{1}{Q^2} \mathbb{E} \int_Q \int_Q |\mathcal{G}g(t, x) - \mathcal{G}g(s, y)|^q dt dx ds dy \\
&\leq N \left(\left\| \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{q/2} \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right. \\
&\quad + \left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^{q_0} \nu(dz) \right\|_{L^{\tilde{\kappa}}(\Omega)}^{q/q_0} \\
&\quad \left. + \left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right), \tag{3.3}
\end{aligned}$$

where $N = N(T, q, \varphi)$. Since the operator \mathcal{G} is translation invariant with respect to x , i.e.

$$\mathcal{G}g(\cdot, \cdot)(t, x + x_0) = \mathcal{G}g(\cdot, x_0 + \cdot)(t, x),$$

we may assume that $x_0 = 0$. We divide the left hand side of (3.3) into two parts. Indeed,

$$\begin{aligned}
&\frac{1}{Q^2} \mathbb{E} \int_Q \int_Q |\mathcal{G}g(t, x) - \mathcal{G}g(s, y)|^q dt dx ds dy \\
&\leq \frac{2}{Q} \mathbb{E} \int_Q |\mathcal{G}g_1(t, x)|^q dt dx ds dy \\
&\quad + \frac{1}{Q^2} \mathbb{E} \int_Q \int_Q |\mathcal{G}g_2(t, x) - \mathcal{G}g_2(s, y)|^q dt dx ds dy \\
&=: J_1 + J_2,
\end{aligned}$$

where

$$g_1(t, x, z) = I_{((t_0-2c^\gamma) \vee 0, t_0+2c^\gamma) \times B_{2c} \times Z}(t, x, z)g(t, x, z), \quad g_2 = g - g_1.$$

Estimate of J_1 . Since $Q \subset \mathcal{O}_T$, it holds that $t_0 - c^\gamma \geq 0$ and thus

$$(t_0 - c^\gamma, t_0 + c^\gamma) \subset (t_0 - 2c^\gamma) \vee 0, t_0 + 2c^\gamma$$

and g vanishes on

$$\left[((t_0 - 2c^\gamma) \vee 0, t_0 + 2c^\gamma) \times B_{2c}^c \times Z \right] \cup \left[(0, (t_0 - 2c^\gamma) \vee 0) \times \mathbb{R}^d \times Z \right].$$

It follows from Lemma 3.1 with $a = (t_0 - 2c^\gamma) \vee 0$ and $b = t_0 + 2c^\gamma$ that

$$J_1 \leq N \left\| \int_Z |g(\cdot, \cdot, z)|_{L^\infty(\mathcal{O}_T)}^{q_0} \nu(dz) \right\|_{L^{\frac{\kappa}{\kappa-q_0}}(\Omega)}^{q/q_0}. \quad (3.4)$$

Estimate of J_2 . If $t_0 \leq 2c^\gamma$, we apply Lemma 3.2 with $a = t_0 - c^\gamma$ and $b = t_0 + c^\gamma$. In this case, one can easily check that $bc^{-\gamma} \leq 3$ and

$$g_2 = 0 \quad \text{on} \quad \left[(0, t_0 + 2c^\gamma) \times B_{2c} \times Z \right].$$

(3.1) of Lemma 3.2 yields that

$$\begin{aligned} J_2 &\leq N \left(\left\| \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{q/2} \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right. \\ &\quad \left. + \left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right) \end{aligned} \quad (3.5)$$

On the other hand, if $t_0 > 2c^\gamma$, we apply Lemma 3.3 with $a = t_0 - c^\gamma$ and $b = t_0 + c^\gamma$. In this case, one can easily check that $3a > b$ and

$$g_2 = 0 \quad \text{on} \quad \left[(t_0 - 2c^\gamma, t_0 + 2c^\gamma) \times B_{2c} \times Z \right].$$

Moreover, by using the nondecreasing of φ , we have

$$\begin{aligned} &\sup_{t_0 \in \mathbb{R}_+, c > 0} \Phi(t_0 - c^\gamma, t_0 + c^\gamma, c) \\ &= \sup_{t_0 \in \mathbb{R}_+, c > 0} \left\{ [\varphi(2)]^{q/q_0} + [\varphi((b-a)c^{-\gamma})]^{q/q_0} + [\varphi(2^{1+1/\gamma}c(b-a)^{-1/\gamma})]^{q/q_0} \right\} \left| \begin{cases} a = t_0 - c^\gamma \\ b = t_0 + c^\gamma \end{cases} \right. \\ &< \infty. \end{aligned}$$

(3.2) implies that

$$\begin{aligned} J_2 &\leq N \left(\left\| \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{q/2} \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right. \\ &\quad \left. + \left\| \int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right\|_{L^{\frac{\kappa}{\kappa-q}}} \right). \end{aligned} \quad (3.6)$$

Combining (3.4), (3.5) and (3.6), we obtain (3.3). The proof of Theorem 2.1 is complete. \square

Remark 3.1 *In this paper, we only consider the simply case. Actually, one can use the similar method and Kunita's second inequality (see Page 268 in [1]) to deal with the following case*

$$\begin{aligned} \mathcal{G}\hat{g}(t, x) &= \int_0^t \int_{\mathbb{R}^d} K(t, s, x - y) h(s, y) dy dW(s) \\ &\quad + \int_0^t \int_Z \int_{\mathbb{R}^d} K(t, s, x - y) g(s, y, z) dy \tilde{N}(dz, ds), \end{aligned}$$

where W and \tilde{N} is a Wiener process and a compensated Poisson measure, respectively. Also see [14] for this case.

4 Applications

In this section, applying Theorem 2.1, we obtain the BMO estimate of the following stochastic singular integral operator

$$\mathcal{G}g(t, x) = \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}^m} \int_{\mathbb{R}^d} K(t, s, x - y) g^k(s, y) dy z \tilde{N}_k(dz, ds), \quad (4.1)$$

where $K(t, s, x) = \nabla^\beta p(t, s, x)$ and $p(t, s, x)$ is the heat kernel of the equation

$$\partial_t u = \Delta^{\frac{\alpha}{2}} u.$$

The fractional derivative of spatial variable is understood in sense of Fourier transform. It is easy to see that

$$\sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}^m} \int_{\mathbb{R}^d} K(t, s, x - y) g^k(s, y) dy z \tilde{N}_k(dz, ds)$$

is the fundamental solution to the following equation

$$du_t(x) = \Delta^{\frac{\alpha}{2}} u_t(x) dt + \sum_{k=1}^{\infty} \int_{\mathbb{R}^m} g^k(t, x) z \tilde{N}_k(dz, dt), \quad u_0 = 0, \quad 0 \leq t \leq T, \quad (4.2)$$

where $\int_{\mathbb{R}^m} z \tilde{N}_k(t, dz) =: Y_t^k$ are independent m -dimensional pure jump Lévy processes with Lévy measure of ν^k . Indeed, one can use the method of [9] (see the proof of Lemma 6.1) to prove the above result. On the other hand, Kim-Kim [12] considered the general case. We only recall the results concerned with this paper. In section 3 of [12], Kim-Kim studied the following linear equation (see Page 3935 of [12]):

$$du = (a(\omega, t) \Delta^{\frac{\alpha}{2}} u + f) dt + \sum_{i=1}^{\infty} h^i dW_t^i + \sum_{k=1}^{\infty} \sum_{j=1}^m g^{k,j} \cdot dY_t^{k,j}, \quad u(0) = u_0, \quad (4.3)$$

where $h = (h^1, h^2, \dots)$, W_t^k is independent one-dimensional Wiener processes and $Y_t^k := \int_{\mathbb{R}^m} z \tilde{N}_k(t, dz)$. Note that Y_t^k are independent m -dimensional pure jump Lévy processes with Lévy measure of ν^k . For any $q, k = 1, 2, \dots$, denote

$$\hat{c}_{k,q} := \left(\int_{\mathbb{R}^m} |z|^q \nu^k(dz) \right)^{\frac{1}{q}}.$$

Fix $p \in [2, \infty)$ and set $\hat{c}_k := \hat{c}_{k,2} \vee \hat{c}_{k,p}$. Assume that

$$\hat{c} := \sup_{k \geq 1} \hat{c}_k < \infty. \quad (4.4)$$

Let \mathcal{P} be the predictable σ -field generated by $\{\mathcal{F}_t, t \geq 0\}$ and $\bar{\mathcal{P}}$ be the completion of \mathcal{P} with respect to $dP \times dt$. For $\eta \in \mathbb{R}$, define $\mathbb{H}_p^\eta(T) := L^p(\Omega \times [0, T], \bar{\mathcal{P}}, H_p^\eta)$, that is, $\mathbb{H}_p^\eta(T)$ is the set of all $\bar{\mathcal{P}}$ -measurable processes $u : \Omega \times [0, T] \mapsto H_p^\eta$ so that

$$\|u\|_{\mathbb{H}_p^\eta(T)} := \left(\mathbb{E} \int_0^T \|u(\omega, t, \cdot)\|_{H_p^\eta}^p dt \right)^{1/p} < \infty,$$

where $H_p^\eta(\mathbb{R}^d) := \{u : D^{\mathbf{n}} u \in L^p(\mathbb{R}^d), |\mathbf{n}| \leq \eta\}$ for $\eta = 1, 2, \dots$. And when η is not an integer, $H_p^\eta(\mathbb{R}^d)$ is defined by Fourier transform.

For ℓ_2 -valued $\bar{\mathcal{P}}$ -measurable processes $g = (g^1, g^2, \dots)$, we write $g \in \mathbb{H}_p^\eta(T, \ell_2)$ if

$$\begin{aligned} \|g\|_{\mathbb{H}_p^\eta(T, \ell_2)} &:= \left(\mathbb{E} \int_0^T \|g(\omega, t, \cdot)\|_{H_p^\eta(T, \ell_2)}^p dt \right)^{1/p} \\ &= \left(\mathbb{E} \int_0^T \| (1 - \Delta)^{\eta/2} g(\omega, t, \cdot) \|_{\ell_2}^p dt \right)^{1/p} < \infty. \end{aligned}$$

Lastly, we define

$$\begin{aligned} \|u\|_{\mathcal{H}_p^{\eta+\alpha}(T)} &:= \|u\|_{\mathbb{H}_p^{\eta+\alpha}(T)} + \|f\|_{\mathbb{H}_p^{\eta+\alpha}(T)} + \|h\|_{\mathbb{H}_p^{\eta+\alpha/2}(T, \ell_2)} \\ &\quad + \sum_{j=1}^m \|g^{\cdot j}\|_{\mathbb{H}_p^{\eta+\alpha/2}(T, \ell_2)} + \|u(0)\|_{U_p^{\eta+\alpha-\alpha/p}}, \end{aligned}$$

where $\|u(0)\|_{U_p^{\eta+\alpha-\alpha/p}} := \left(\mathbb{E}[\|u_0\|_{H_p^\eta}^p] \right)^{1/p}$.

Proposition 4.1 [12, Theorem 3.6] *Suppose (4.4) holds. Then for any $f \in \mathcal{H}_p^\eta(T)$, $h \in \mathbb{H}_p^{\eta+\alpha/2}(T, \ell_2)$, $g^{\cdot j} \in \mathbb{H}_p^{\eta+\alpha-\alpha/p}(T, \ell_2)$, $1 \leq j \leq m$ and $u_0 \in U_p^{\eta+\alpha-\alpha/p}$, Eq. (4.3) has a unique solution u in $\mathcal{H}_p^{\eta+\alpha}$, and for this solution*

$$\begin{aligned} \|u\|_{\mathcal{H}_p^{\eta+\alpha}(t)} &\leq N(p, T, a) \left(\|f\|_{\mathbb{H}_p^\eta(t)} + \|h\|_{\mathbb{H}_p^{\eta+\alpha/2}(t, \ell_2)} \right. \\ &\quad \left. + \sum_{j=1}^m \|g^{\cdot j}\|_{\mathbb{H}_p^{\eta+\alpha-\alpha/p}(t, \ell_2)} + \|u(0)\|_{U_p^{\eta+\alpha-\alpha/p}} \right) \end{aligned}$$

for every $t \leq T$.

In order to investigate the BMO estimate of the solution, we recall some properties of kernel $p(t, s, x)$ (see [2, 3, 5, 7] for more details).

- for any $t > 0$,

$$\|p(t, \cdot)\|_{L^1(\mathbb{R}^d)} = 1 \text{ for all } t > 0.$$

- $p(t, x, y)$ is C^∞ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ for each $t > 0$;
- for $t > 0$, $x, y \in \mathbb{R}^d$, $x \neq y$, the sharp estimate of $\widehat{p}(t, x)$ is

$$p(t, x, y) \approx \min \left(\frac{t}{|x - y|^{d+\alpha}}, t^{-d/\alpha} \right);$$

- for $t > 0$, $x, y \in \mathbb{R}^d$, $x \neq y$, the estimate of the first order derivative of $\widehat{p}(t, x)$ is

$$|\nabla_x p(t, x, y)| \approx |y - x| \min \left\{ \frac{t}{|y - x|^{d+2+\alpha}}, t^{-\frac{d+2}{\alpha}} \right\}. \quad (4.5)$$

The notation $f(x) \approx g(x)$ means that there is a number $0 < C < \infty$ independent of x , i.e. a constant, such that for every x we have $C^{-1}f(x) \leq g(x) \leq Cf(x)$. The estimate (4.5) for the first order derivative of $p(t, x)$ was derived in [2, Lemma 5]. Xie et al. [17] the estimate of the m -th order derivative of $p(t, x)$ by induction.

Proposition 4.2 [17, Lemma 2.1] *For any $m \geq 0$, we have*

$$\partial_x^m p(t, x) = \sum_{n=0}^{n=\lfloor \frac{m}{2} \rfloor} C_n |x|^{m-2n} \min \left\{ \frac{t}{|x|^{d+\alpha+2(m-n)}}, t^{-\frac{d+2(m-n)}{\alpha}} \right\}, \quad (4.6)$$

where $\lfloor \frac{m}{2} \rfloor$ means the largest integer that is less than $\frac{m}{2}$.

Next, we claim that the kernel $\nabla^{\frac{\alpha}{q_0}} p(t, s, x)$, $q_0 \geq 2$, satisfies the **Assumption 2.3** with $\gamma = \alpha$ and $\kappa = \infty$.

Lemma 4.1 *Let $\beta = \frac{\alpha}{q_0}$. The following estimates hold.*

(i) *For any $t > \lambda > 0$ and $c > 0$,*

$$\int_{\lambda}^t \left| \int_{|x| \geq c} |\nabla^{\beta} p(t, r, x)| dx \right|^{q_0} dr \leq N \left([(t - \lambda)c^{-\alpha}]^{q_0+1} + [(t - \lambda)c^{-\alpha}] \right);$$

(ii) *For any $t > s > \lambda > 0$,*

$$\int_0^{\lambda} \left(\int_{\mathbb{R}^d} |\nabla^{\beta} p(t, r, x) - \nabla^{\beta} p(s, r, x)| dx \right)^{q_0} dr \leq N [(t - s)(t \wedge s - \lambda)^{-1}]^{q_0};$$

(iii) *For any $s > \lambda \geq 0$ and $h \in \mathbb{R}^d$,*

$$\int_0^{\lambda} \left(\int_{\mathbb{R}^d} |\nabla^{\beta} p(s, r, x + h) - \nabla^{\beta} p(s, r, x)| dx \right)^{q_0} dr \leq N \varphi(|h|(s - \lambda)^{-1/\alpha}).$$

Proof. Note that $\beta = \frac{\alpha}{q_0} < 2$. By using Proposition 4.2, we have if $c > (t - r)^{\frac{1}{\alpha}}$,

$$\begin{aligned} & \int_{\lambda}^t \left| \int_{|x| \geq c} |\nabla^{\beta} p(t, r, x)| dx \right|^{q_0} dr \\ & \leq N \int_{\lambda}^t \left| \int_{|x| \geq c} |x|^{\beta} \frac{t - r}{|x|^{d+\alpha+2\beta}} dx \right|^{q_0} dr \\ & \leq N \int_{\lambda}^t \left| \int_c^{\infty} |x|^{\beta} \cdot |x|^{d-1} \frac{t - r}{|x|^{d+\alpha+2\beta}} d|x| \right|^{q_0} dr \\ & = N c^{-\alpha(q_0+1)} \int_{\lambda}^t (t - r)^{q_0} dr \\ & \leq N [(t - \lambda)c^{-\alpha}]^{q_0+1}. \end{aligned}$$

When $c \leq (t - r)^{\frac{1}{\alpha}}$, we have $(t - r)^{-1} \leq c^{-\alpha}$

$$\begin{aligned} & \int_{\lambda}^t \left| \int_{|x| \geq c} |\nabla^{\beta} p(t, r, x)| dx \right|^{q_0} dr \\ & \leq N \int_{\lambda}^t \left(\int_{(t-r)^{\frac{1}{\alpha}}}^{\infty} |x|^{\beta} \cdot |x|^{d-1} \frac{t - r}{|x|^{d+\alpha+2\beta}} d|x| \right. \\ & \quad \left. + \int_c^{(t-r)^{\frac{1}{\alpha}}} |x|^{\beta} \cdot |x|^{d-1} (t - r)^{-\frac{d+2\beta}{\alpha}} d|x| \right)^{q_0} dr \\ & \leq N \int_{\lambda}^t \left(\int_c^{\infty} |x|^{\beta} \cdot |x|^{d-1} \frac{t - r}{|x|^{d+\alpha+2\beta}} d|x| \right. \\ & \quad \left. + \int_0^{(t-r)^{\frac{1}{\alpha}}} |x|^{\beta} \cdot |x|^{d-1} (t - r)^{-\frac{d+2\beta}{\alpha}} d|x| \right)^{q_0} dr \\ & \leq N c^{-\alpha(q_0+1)} \int_{\lambda}^t (t - r)^{q_0} dr + N c^{-\alpha} \int_{\lambda}^t dr \\ & \leq N [(t - \lambda)c^{-\alpha}]^{q_0+1} + [(t - \lambda)c^{-\alpha}]. \end{aligned}$$

Hence we obtain the first estimate.

When $\alpha + \frac{\alpha}{q_0} < 2$, $\lfloor \frac{\alpha + \alpha/q_0}{2} \rfloor = 0$. Using the fact that $\partial_t p = \Delta^{\alpha/2} p$, $\beta q_0 = 1$ and Proposition 4.2, we get

$$\begin{aligned}
& \int_0^\lambda \left(\int_{\mathbb{R}^d} |\nabla^\beta p(t, r, x) - \nabla^\beta p(s, r, x)| dx \right)^{q_0} dr \\
& \leq (t-s)^{q_0} \int_0^\lambda \left(\int_{\mathbb{R}^d} |\nabla^{\alpha+\beta} p(\xi-r, x)| dx \right)^{q_0} dr \\
& \leq N(t-s)^{q_0} \int_0^\lambda \left(\int_0^{(\xi-r)^{\frac{1}{\alpha}}} |x|^{\alpha+\beta} |x|^{d-1} (\xi-r)^{-\frac{d+2\alpha+2\beta}{\alpha}} d|x| \right. \\
& \quad \left. + \int_{(\xi-r)^{\frac{1}{\alpha}}}^\infty |x|^{\alpha+\beta} |x|^{d-1} \frac{\xi-r}{|x|^{d+3\alpha+2\beta}} d|x| \right)^{q_0} dr \\
& \leq N(t-s)^{q_0} \int_0^\lambda (\xi-r)^{-q_0-1} dr \\
& \leq N[(t-s)(t \wedge s - \lambda)^{-1}]^{q_0},
\end{aligned}$$

where $\xi = \theta t + (1-\theta)s$, $\theta \in [0, 1]$.

Since $q_0 \geq 2$ and $0 \leq \alpha \leq 2$, we have $\alpha + \frac{\alpha}{q_0} < 4$. When $2 \leq \alpha + \frac{\alpha}{q_0} < 4$, we have

$$\begin{aligned}
& \int_0^\lambda \left(\int_{\mathbb{R}^d} |\nabla^\beta p(t, r, x) - \nabla^{\frac{\alpha}{q_0}} p(s, r, x)| dx \right)^{q_0} dr \\
& \leq (t-s)^{q_0} \int_0^\lambda \left(\int_{\mathbb{R}^d} |\nabla^{\alpha+\beta} p(\xi-r, x)| dx \right)^{q_0} dr \\
& \leq N(t-s)^{q_0} \int_0^\lambda \left(\int_0^{(\xi-r)^{\frac{1}{\alpha}}} |x|^{\alpha+\beta} |x|^{d-1} (\xi-r)^{-\frac{d+2\alpha+2\beta}{\alpha}} d|x| \right. \\
& \quad \int_0^{(\xi-r)^{\frac{1}{\alpha}}} |x|^{\alpha+\beta-2} |x|^{d-1} (\xi-r)^{-\frac{d+2\alpha+2\beta-2}{\alpha}} d|x| \\
& \quad + \int_{(\xi-r)^{\frac{1}{\alpha}}}^\infty |x|^{\alpha+\beta} |x|^{d-1} \frac{\xi-r}{|x|^{d+3\alpha+2\beta}} d|x| \\
& \quad \left. + \int_{(\xi-r)^{\frac{1}{\alpha}}}^\infty |x|^{\alpha+\beta-2} |x|^{d-1} \frac{\xi-r}{|x|^{d+3\alpha+2\beta-2}} d|x| \right)^{q_0} dr \\
& \leq N(t-s)^{q_0} \int_0^\lambda (\xi-r)^{-q_0-1} dr \\
& \leq N[(t-s)(t \wedge s - \lambda)^{-1}]^{q_0},
\end{aligned}$$

where $\xi = \theta t + (1-\theta)s$, $\theta \in [0, 1]$. Thus we obtain the second estimate.

For the last estimate (iii), noting that $1 + \beta \leq 2$, we have for $1 + \beta < 2$

$$\begin{aligned}
& \int_0^\lambda \left(\int_{\mathbb{R}^d} |\nabla^\beta p(s, r, x + h) - \nabla^\beta p(s, r, x)| dx \right)^{q_0} dr \\
& \leq N \int_0^\lambda h^{q_0} \left(\int_{\mathbb{R}^d} |\nabla^{1+\beta} p(s, r, x + \theta h)| dx \right)^{q_0} dr \\
& \leq N \int_0^\lambda h^{q_0} \left(\int_0^{(s-r)^{\frac{1}{\alpha}}} |x|^{1+\beta} \cdot |x|^{d-1} (s-r)^{-\frac{d+2+2\beta}{\alpha}} d|x| \right. \\
& \quad \left. + \int_{(s-r)^{\frac{1}{\alpha}}}^\infty |x|^{1+\beta} \cdot |x|^{d-1} \frac{s-r}{|x|^{d+\alpha+2+2\beta}} d|x| \right)^{q_0} dr \\
& \leq N [h(s-\lambda)^{-1}]^{q_0},
\end{aligned}$$

where $\theta \in [0, 1]$. When $1 + \beta = 2$, similar the case (ii), one can get the same estimate. The proof of Lemma is complete. \square

It follows from the Proposition 4.1 that $\nabla^\beta p(t, s, x)$ satisfies the **Assumption 2.4**. By using Theorem 2.1, we have the following result.

Theorem 4.1 *Let $q_0 \geq 2$. Suppose (4.4) with $p \geq q_0$ holds. Then for any $g \in \mathbb{H}_p^{\eta+\alpha-\alpha/p}(T, \ell_2)$, Eq. (4.2) has a unique solution u in $\mathcal{H}_p^{\eta+\alpha}$ ($\eta \in \mathbb{R}$), and for this solution*

$$\|u\|_{\mathcal{H}_p^{\eta+\alpha}(t)} \leq N(p, T) \|g\|_{\mathbb{H}_p^{\eta+\alpha-\alpha/p}(t, \ell_2)}$$

for every $t \leq T$.

Moreover, we have for $q \in [2, q_0]$

$$[\nabla^\beta u]_{\mathbb{BMO}(T, q)} \leq N \hat{c} \left(\mathbb{E}[\|g\|_{\ell_2}^{q_0}] \right)^{q/q_0},$$

where $\beta = \alpha/q_0$ and \hat{c} is defined as in (4.4).

When the Lévy noise is replaced by Brownian motion in (4.2), i.e.,

$$du_t(x) = \Delta^{\frac{\alpha}{2}} u_t(x) dt + \sum_{k=1}^\infty h^k(t, x) dW_t^k, \quad u_0 = 0, \quad 0 \leq t \leq T, \quad (4.7)$$

where W_t^k are independent one-dimensional Wiener processes. Denote $h = (h^1, h^2, \dots)$.

Similar to Lemma 4.1, one can prove $\nabla^{\frac{\alpha}{2}} p(t, s, x)$ satisfies the **Assumption 2.1**. From Proposition 4.1, we know that **Assumption 2.2** holds for $\nabla^{\frac{\alpha}{2}} p(t, s, x)$. Thus we can get the following result.

Theorem 4.2 *Suppose that $h \in L^p(T, \ell_2)$, there exists a uniqueness solution u in $\mathcal{H}_p^{\eta+\alpha}$ ($\eta \in \mathbb{R}$), and for this solution*

$$\|u\|_{\mathcal{H}_p^{\eta+\alpha}(t)} \leq N(p, T) \|h\|_{\mathbb{H}_p^{\eta+\alpha/2}(t, \ell_2)}$$

for every $t \leq T$.

Moreover, we have for any $q \in (0, p]$

$$[\nabla^{\frac{\alpha}{2}} u]_{\mathbb{BMO}(T, q)} \leq N \left(\mathbb{E}[\|h\|_{\ell_2}^p] \right)^{1/p}.$$

Remark 4.1 1. In Lemma 4.1, the second part (ii) is essential. From the proof of Theorem 2.1, the bound of the BMO norm can be controlled by the function φ and some norm of g , where the bound of the function φ depends on the choice of scale of time and space. In second part (ii), we must prove that the left hand side of (ii) can be controlled by the function of $(t-s)(t \wedge s - \lambda)^{-1}$. Only in this form, the left hand side of (ii) can be controlled by a constant.

2. Particularly, taking $q_0 = 2$, we have Lemma 4.1 holds for $\nabla^{\frac{\alpha}{2}}p(t, s, x)$. Hence we have Theorem 4.2. Noting that if $\alpha = 2$, Theorem 4.2 becomes [9, Theorem 3.4]. Thus we generalize the result of [9].

5 Discussion

In this section, we give another proof of Theorem 2.1 under some assumptions on g . Similarly, one can give another proof of [9, Theorem 2.4] under the same assumptions on g . Firstly, let us recall the proofs of Theorem 2.1 and [9, Theorem 2.4]. The reason why we divide the interval $(0, s)$ into two parts $(0, \frac{3a-b}{2})$ and $(\frac{3a-b}{2}, s)$ in proof of Lemma 3.3 is the singularity of K at time t . In order to see it clearly, we look at the Section 4 and recall that for any $t > \lambda > 0$ and $c > 0$,

$$\int_{\lambda}^t \left| \int_{|x| \geq c} |\nabla^{\beta} p(t, r, x)| dx \right|^{q_0} dr \leq N \left([(t - \lambda)c^{-\alpha}]^{q_0+1} + [(t - \lambda)c^{-\alpha}] \right).$$

Note that if we choose $c = 0$, then the above integral will be infinity. Indeed, direct calculations show that

$$\int_{\lambda}^t \left| \int_{\mathbb{R}^d} |\nabla^{\beta} p(t, r, x)| dx \right|^{q_0} dr \approx N \int_{\lambda}^t (t - r)^{-1} dr = \infty.$$

Obviously, the singularity of $\nabla^{\beta}p$ appears at t . But $p \in L^1(\mathbb{R}^d)$, thus a natural question appears: when the singularity of p does not appear at t , is there another proof? Moreover, it is easy to see that the derivative of p deduces the singularity of $\nabla^{\beta}p$ at t . In this section, we first give a similar theorem to Theorem 2.1 under different assumptions. Then as an application, we use the method of integration by part to deal with the derivative of p and obtain the BMO estimate by direct calculation.

Theorem 5.1 Assume that the kernel function is a deterministic function and satisfies that for all $t \geq r \geq 0$,

$$\int_0^t \int_{\mathbb{R}^d} |K(t, r, x)| dx dr \leq N(T).$$

Assume further that there exists a positive constant $q_0 > 2$ such that

$$\mathbb{E} \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^{\infty}(\mathcal{O}_T)}^{\varpi} \nu(dz) \right)^{\frac{q_0}{2}} < \infty, \quad \varpi = 2 \text{ or } q_0.$$

Then for any $q \in (0, q_0]$, one has

$$\begin{aligned} [\mathcal{G}g]_{\text{BMO}(T, q)} &\leq N \mathbb{E} \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^{\infty}(\mathcal{O}_T)}^2 \nu(dz) \right)^{\frac{q}{2}} \\ &\quad + \mathbb{E} \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^{\infty}(\mathcal{O}_T)}^q \nu(dz) \right), \end{aligned}$$

where $N = N(N_0, d, q, q_0, T)$.

Proof. It suffices to prove that for each

$$Q = Q_c(t_0, x_0) := (t_0 - c^\gamma, t_0 + c^\gamma) \times B_c(x_0) \subset \mathcal{O}_T, \quad c > 0, t_0 > 0,$$

we have

$$\begin{aligned} & \frac{1}{Q^2} \mathbb{E} \int_Q \int_Q |\mathcal{G}g(t, x) - \mathcal{G}g(s, y)|^q dt dx ds dy \\ & \leq N \mathbb{E} \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{\frac{q}{2}} + \mathbb{E} \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right), \end{aligned} \quad (5.1)$$

where $N = N(T, q, \varphi)$. Since the operator \mathcal{G} is translation invariant with respect to x , we may assume that $x_0 = 0$. Kunita's first inequality implies that

$$\begin{aligned} \mathbb{E} |\mathcal{G}g(t, x)|^q & \leq \mathbb{E} \left(\int_0^t \int_Z \left| \int_{\mathbb{R}^d} k(t-r, y) g(r, x-y, z) dy \right|^2 \nu(dz) dr \right)^{q/2} \\ & \quad + \mathbb{E} \left(\int_0^t \int_Z \left| \int_{\mathbb{R}^d} k(t-r, y) g(r, x-y, z) dy \right|^q \nu(dz) dr \right) \\ & \leq \mathbb{E} \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \times \int_0^t \left| \int_{\mathbb{R}^d} k(t-r, y) dy \right|^2 dr \right)^{\frac{q}{2}} \\ & \quad + \mathbb{E} \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \times \int_0^t \left| \int_{\mathbb{R}^d} k(t-r, y) dy \right|^q dr \right) \\ & \leq N(T) \mathbb{E} \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{\frac{q}{2}} \\ & \quad + \mathbb{E} \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right) \\ & < \infty. \end{aligned}$$

Thus we have

$$\begin{aligned} & \frac{1}{Q^2} \mathbb{E} \int_Q \int_Q |\mathcal{G}g(t, x) - \mathcal{G}g(s, y)|^q dt dx ds dy \\ & \leq \frac{2}{Q} \mathbb{E} \int_Q |\mathcal{G}g(t, x)|^q dt dx \\ & \leq N(T) \mathbb{E} \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{\frac{q}{2}} \\ & \quad + \mathbb{E} \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right), \end{aligned}$$

which implies (5.1) holds. The proof of Theorem 5.1 is complete. \square

As an application, for simplicity, we consider the following stochastic evolution equation

$$du = \Delta u dt + \int_Z g(t, x, z) \tilde{N}(dt, dz) \quad u(0, x) = 0. \quad (5.2)$$

It is easy to check that the solution of (5.2) is

$$u(t, x) = \int_0^t \int_Z \int_{\mathbb{R}^d} K(t-r, y) g(r, y, z) dy d\tilde{N}(dr, dz).$$

It follows the properties of heat kernel that

$$\int_{\mathbb{R}^d} |K(t, r, x)| dx = 1 \quad \text{for all } t > r > 0.$$

Applying Theorem 5.1, we have

Theorem 5.2 Assume that there exists a positive constant $q_0 > 2$ such that

$$\mathbb{E} \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^\varpi \nu(dz) \right)^{\frac{q_0}{2}} < \infty, \quad \varpi = 2 \text{ or } q_0.$$

Then for any $q \in (0, q_0]$, one has

$$\begin{aligned} [u]_{\mathbb{BMO}(T, q)} &\leq N \mathbb{E} \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{\frac{q}{2}} \\ &\quad + \mathbb{E} \left(\int_Z \|g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right), \end{aligned}$$

where $N = N(N_0, d, q, q_0, T)$. Moreover, if we further assume that

$$\mathbb{E} \left(\int_Z \|\nabla_x g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^\varpi \nu(dz) \right)^{\frac{q_0}{2}} < \infty, \quad \varpi = 2 \text{ or } q_0.$$

Then for any $q \in (0, q_0]$, one has

$$\begin{aligned} [\nabla u]_{\mathbb{BMO}(T, q)} &\leq N \mathbb{E} \left(\int_Z \|\nabla_x g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^2 \nu(dz) \right)^{\frac{q}{2}} \\ &\quad + \mathbb{E} \left(\int_Z \|\nabla_x g(\cdot, \cdot, z)\|_{L^\infty(\mathcal{O}_T)}^q \nu(dz) \right), \end{aligned}$$

where $N = N(N_0, d, q, q_0, T)$ and $\nabla_x g = \nabla_x g(t, \cdot, z)$.

Proof. Denote $u(t, x) = \mathcal{G}g(t, x)$. Noting that

$$\nabla_x \mathcal{G}g(t, x) = \int_0^t \int_Z \int_{\mathbb{R}^d} k(t-r, y) \nabla_x g(r, x-y, z) dy \tilde{N}(dr, dz).$$

Then similar to the proof of Theorem 5.1, one can get the desired result. \square

Remark 5.1 Comparing with the proofs of Theorems 2.1 and 5.1, we find if we assume the function g has high regularity, then the proof of BMO estimate will be very simple. The proof of Theorem 4.1 will be also simple if we improve the regularity of g .

If $g \equiv 0$, then $u \equiv 0$. That is to say, the noise has effect on the regularity of the solutions.

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